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Some equitably 3-colourable cycle decompositions of complete equipartite graphs

James Lefevre, Mary Waterhouse

Department of Mathematics, The University of Queensland, Qld 4072, Australia

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Abstract

Let G be a graph in which each vertex has been coloured using one of k colours, say $c_1, c_2, \ldots, c_k$. If an m-cycle C in G has n_i vertices coloured c_i, $i = 1, 2, \ldots, k$, and $|n_i - n_j| \leqslant 1$ for any $i, j \in \{1, 2, \ldots, k\}$, then C is said to be equitably k-coloured. An m-cycle decomposition $\mathscr{C}$ of a graph G is equitably k-colourable if the vertices of G can be coloured so that every m-cycle in $\mathscr{C}$ is equitably k-coloured. For $m = 3$, 4 and 5 we completely settle the existence question for equitably 3-colourable m-cycle decompositions of complete equipartite graphs.

Keywords: Graph colourings; Cycle decomposition; Complete equipartite graphs

1. Introduction

Let $K_{n_1,n_2,\ldots,n_p}$ denote the complete multipartite graph with n_i vertices in part i, for $1 \leqslant i \leqslant p$. If each of the p parts contains n vertices, the graph is said to be *complete equipartite* and is denoted $K_{p(n)}$. The graph $K_{p(1)}$ is isomorphic to the complete graph on p vertices, denoted K_p.

A *pairwise balanced design* with parameters v, K and λ (sometimes written PBD(v, K, λ)) is a pair $(V, \mathscr{B})$ such that V is a v-set of elements and $\mathscr{B}$ is a collection of subsets of V (where each member $B \in \mathscr{B}$ is called a *block* of the design). Furthermore, for each block $B \in \mathscr{B}$, $|B| \in K$, and every unordered pair of elements in V occurs together in precisely λ blocks of $\mathscr{B}$.

E-mail address: maw@maths.uq.edu.au (M. Waterhouse).

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A *group divisible design* with parameters K, λ, M and v (sometimes written GDD[$K, \lambda, M; v$]) is a triple $(V, \Gamma, \mathscr{B})$ such that V is a v-set of elements, $\Gamma = \{G_1, G_2, \ldots\}$ is a partition of V (where each class of the partition is called a *group* of the design), and $\mathscr{B}$ is a collection of subsets of V (where each member $B \in \mathscr{B}$ is called a *block* of the design). Furthermore, for each group $G_i \in \Gamma$, $|G_i| \in M$, for each block $B \in \mathscr{B}$, $|B| \in K$, and for all $x, y \in V$, x and y occur together in precisely λ blocks of $\mathscr{B}$ if they do not appear together in any group of Γ, and x and y appear in no blocks of $\mathscr{B}$ otherwise.

Let G and $H_1, H_2, \ldots, H_n$ be graphs. The set $\mathscr{H} = \{H_1, H_2, \ldots, H_n\}$ is said to be a *decomposition* of G if $\mathscr{H}$ partitions the edge set of G. If H_i is isomorphic to H, for $1 \leqslant i \leqslant n$, then $\mathscr{H}$ is said to be an H-decomposition of G.

In this paper, G is usually a complete equipartite graph and H is usually an m-cycle, where $m \in \{3, 4, 5\}$. An m-cycle decomposition of a graph G is possible only if m divides the total number of edges in G and each vertex in G has even degree.

A decomposition of $K_{n_1,n_2,\ldots,n_p}$ into copies of the complete graph on v vertices can be thought of as a GDD[$v, 1, M; \sum_{i=1}^{p} n_i$], where $n_i \in M$, for $1 \leqslant i \leqslant p$. Indeed, a 3-cycle decomposition of $K_{n_1,n_2,\ldots,n_p}$ can be thought of as a GDD[$3, 1, M; \sum_{i=1}^{p} n_i$], where $n_i \in M$, for $1 \leqslant i \leqslant p$.

When m is odd, it is impossible to find an m-cycle decomposition of any bipartite graph. Generally speaking, the problem of finding an odd-length cycle decomposition of $K_{n_1,n_2,\ldots,n_p}$, where $p \geqslant 3$, is difficult and remains an open problem. In [13] Hanani completely settled the existence question for GDD[$3, 1, n; pn$], thus solving the existence question for 3-cycle decompositions of $K_{p(n)}$. Other authors have considered GDDs with all blocks having size three, but groups of varying sizes. In [10] the existence question for GDDs with block size three, x groups of size t and one group of size u is completely settled for all values of x, t and u. The authors of [8] go on to completely settle the existence question for GDDs with block size three, x groups of size t and y groups of size one. Furthermore, in [7] Colbourn determined some necessary conditions for existence of group divisible designs with block size three and proved that these conditions were also sufficient for designs on 60 or fewer elements.

Less attention has been given to m-cycle decompositions of $K_{n_1,n_2,\ldots,n_p}$ when m is odd and greater than three. However, in [3] Billington et al. determined the necessary and sufficient conditions for existence of a 5-cycle decomposition of $K_{p(n)}$. In [6] Cavenagh and Billington completely settle the existence question for 5-cycle decompositions of K_{n_1,n_2,n_3} for the cases $n_1 = n_2$, $n_2 = n_3$ or $n_1 \equiv n_2 \equiv 0 \pmod{10}$. In [4] Cavenagh extended this result to all complete tripartite graphs where all parts are of even size.

When m is even, m-cycle decompositions of the complete bipartite graph are both interesting and useful when constructing m-cycle decompositions of $K_{n_1,n_2,\ldots,n_p}$, for $p \geqslant 3$. In Sotteau's seminal paper [15], it is shown that a $2k$-cycle decomposition of K_{n_1,n_2} exists if and only if $n_1 \equiv n_2 \equiv 0 \pmod{2}$, $n_1 \geqslant 2$, $n_2 \geqslant 2$ and $2k$ divides $n_1 n_2$. In [5] Cavenagh and Billington determine some necessary conditions for existence of an even-length cycle decomposition of $K_{n_1,n_2,\ldots,n_p}$ and prove that these conditions are also sufficient for cycles of lengths 4, 6 and 8.

A *colouring* of an m-cycle decomposition $\mathscr{C}$ of a graph G is an assignment of colours to the vertices of G. A *k-colouring* of $\mathscr{C}$ is a colouring in which k distinct colours are used.

A k-colouring of an m-cycle decomposition $\mathscr{C}$ clearly induces a colouring of each m-cycle in $\mathscr{C}$. If n_i vertices of an m-cycle $C \in \mathscr{C}$ are coloured c_i, then C is said to be *equitably k-coloured* if $|n_i - n_j| \leqslant 1$ for any $i, j \in \{1, 2, \dots, k\}$. An m-cycle decomposition $\mathscr{C}$ is said to be *equitably k-colourable* if it can be k-coloured so that every $C \in \mathscr{C}$ is equitably k-coloured.

It is shown in [1] that an equitably $(m-1)$-coloured m-cycle decomposition of K_v (v odd) or $K_v - F$ (v even and F is a 1-factor of K_v) is possible only if $v \leqslant (m-1)^2$ or $v \leqslant 2(m-1)^2$, respectively. Hence, an equitably 2-coloured 3-cycle decomposition of K_v exists trivially only when $v = 3$. Similarly, an equitably 2-colourable 3-cycle decomposition of $K_v - F$ is possible only if $v \leqslant 8$ and $v \equiv 0, 2 \pmod 6$. An equitably 2-coloured 3-cycle decomposition of $K_2 - F$ exists trivially and it is relatively simple to find such decompositions of both $K_6 - F$ and $K_8 - F$, thus demonstrating that these conditions are also sufficient. However, for $m \geqslant 4$, the problem of determining the values of v for which there exist equitably 2-colourable m-cycle decompositions of K_v or $K_v - F$ is no longer trivial.

Necessary and sufficient conditions for the existence of such decompositions for $m \in \{4, 5, 6\}$ are given in [2]. The existence question for equitably 3-coloured m-cycle decompositions of K_v and $K_v - F$ has also been completely settled for $m \in \{4, 5, 6\}$ in [1].

In [16], the existence question for equitably 2-colourable m-cycle decompositions of $K_{p(n)}$ and $K_{n_1, n_2, \dots, n_p}$ is completely settled for $m \in \{3, 5\}$ and $m \in \{4, 6\}$, respectively. In this paper, we completely settle the existence question for equitably 3-colourable m-cycle decompositions of $K_{p(n)}$ for $m \in \{3, 4, 5\}$. Our main result is given in the following theorem.

Main Theorem

- *There exist equitably* 3-*colourable* 3-*cycle decompositions of* $K_{p(n)}$ *if and only if* $p \in \{1, 3\}$.
- *There exist equitably* 3-*colourable* 4-*cycle decompositions of* $K_{p(n)}$ *if and only if the number of edges in* $K_{p(n)}$ *is divisible by* 4, *each vertex has even degree*, $pn \geqslant 4$, $p \in \{1, 2, 3, 4, 5, 6, 7, 9\}$ *and if* $p = 7$, *then* $n \geqslant 4$.
- *There exist equitably* 3-*colourable* 5-*cycle decompositions of* $K_{p(n)}$ *if and only if the number of edges in* $K_{p(n)}$ *is divisible by* 5, *each vertex has even degree*, $p \neq 2$ *and* $pn \geqslant 5$.

We require some additional notation which we introduce here. The m-cycle with edges $\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_m, x_1\}$ will be denoted by $(x_1, x_2, \dots, x_m)$, or by any cyclic shift of this. Also, we use $C_{p(n)}$ to denote the graph with vertex set $\mathbb{Z}_n \times \{1, 2, \dots, p\}$ and edge set $\{\{(a, i), (b, i+1)\} : a, b \in \mathbb{Z}_n, 1 \leqslant i \leqslant p$, addition modulo $p\}$, where the second number in each ordered pair is used to denote the part to which the vertex belongs. Thus, $C_{p(n)}$ has p parts, each containing n vertices. Elsewhere, we use a_b to denote vertices, where the subscript is used to denote the part to which the vertex belongs. Let G be any graph and let $H_1, H_2, \dots, H_p$ be pairwise disjoint graphs such that $V(G) = V(H_1) \cup V(H_2) \cup \cdots \cup V(H_p)$. We use $G - (H_1 + H_2 + \cdots + H_p)$ to denote the graph formed by taking the graph G and removing the edges of $H_1, H_2, \dots, H_p$. If $H_i \cong H$, for $i = 1, 2, \dots, p$, we denote the graph by $G - (p \times H)$ instead. Finally, in this paper we use the colours black, white and grey, and we say that an edge connecting two vertices of the same colour is a *pure edge* and that an edge connecting two differently coloured vertices is a *mixed edge*.

2. Equitably 3-colourable 3-cycle decompositions

There is only one way to equitably 3-colour a 3-cycle and this cycle contains three distinctly coloured vertices. Hence, the cycle contains only mixed edges and we conclude that an equitably 3-coloured 3-cycle decomposition of $K_{p(n)}$ is possible only if $K_{p(n)}$ does not have any pure edges. Given this constraint, we are in a position to prove Theorem 2.1.

Theorem 2.1. *There exist equitably* 3-*colourable* 3-*cycle decompositions of* $K_{p(n)}$ *if and only if* $p \in \{1, 3\}$.

Proof. Suppose that the decomposition exists. Since no bipartite graph can be decomposed into odd-length cycles, then $p \neq 2$. However, since the decomposition contains no pure edges, then $p = 1$ or 3.

If $p = 1$, the graph has no edges and so a decomposition exists trivially for any n. Suppose then that $p = 3$. Let the vertex set of $K_{3(n)}$ be $\mathbb{Z}_n \times \{1, 2, 3\}$, where the second number in each ordered pair indicates the part to which the vertex belongs. Colour the vertices with three colours such that all the vertices in a part have the same colour. An equitably 3-coloured 3-cycle decomposition of $K_{3(n)}$ is given by developing the starter cycle $((s, 1), (t, 2), (s + t, 3))$, where $s, t \in \mathbb{Z}_n$ and addition is calculated modulo n. $\square$

3. Equitably 3-colourable 4-cycle decompositions

Any equitably 3-coloured 4-cycle contains at most one pure edge. Consequently, an equitably 3-coloured 4-cycle decomposition of $K_{p(n)}$ can exist only if the number of pure edges in $K_{p(n)}$ is at most $\frac{1}{4}$ of the total number of edges. Equivalently, the decomposition can exist only if the number of mixed edges in $K_{p(n)}$ is at least $\frac{3}{4}$ of the total number of edges. We use this, along with Lemmas 3.1–3.9, when, when proving Theorem 3.10. For each existence result involving a multipartite graph, the graph has the obvious vertex partition.

Lemma 3.1 (*Adams et al.* [1]). *There exists an equitably* 3-*coloured* 4-*cycle decomposition of* K_v *if and only if* $v = 9$.

Lemma 3.2 (*Adams et al.* [1]). *There exist equitably* 3-*coloured* 4-*cycle decompositions of* $K_v - F$ *if and only if* $v \in \{4, 6, 8, 10, 12, 18\}$.

For convenience we reproduce an equitably 3-coloured 4-cycle decomposition of $K_{12} - F$ here, since this decomposition is used in a later proof.

Lemma 3.3. *There exists an equitably* 3-*coloured* 4-*cycle decomposition of* $K_{12} - F$.

Proof. Let the vertex set of $K_{12} - F$ be $\bigcup_{i=1}^{3}\{0_i, 1_i, 2_i, 3_i\}$. Colour the vertices with subscript 1 black, those with subscript 2 white and those with subscript 3 grey. The edges in F are $\{0_i, 1_i\}$ and $\{2_i, 3_i\}$, for $i \in \{1, 2, 3\}$. A suitable decomposition of $K_{12} - F$ is given by

$$
\begin{array}{lllll}
(1_2, 1_3, 2_2, 3_1)^*, & (0_2, 1_3, 2_1, 2_3)^*, & (3_2, 0_1, 1_3, 1_1)^*, & (0_2, 3_3, 1_3, 3_1), & (3_2, 2_3, 0_3, 2_1), \\
(2_2, 2_3, 1_1, 2_1), & (0_2, 2_2, 0_3, 0_1), & (0_2, 3_2, 0_3, 1_1), & (0_2, 0_3, 3_3, 2_1), & (1_2, 2_2, 0_1, 2_3), \\
(1_2, 3_2, 3_3, 1_1), & (1_2, 0_3, 3_1, 0_1), & (1_2, 3_3, 0_1, 2_1), & (2_2, 3_3, 3_1, 1_1), & (3_2, 1_3, 2_3, 3_1).
\end{array}
$$

Note. Cycles marked with an asterisk in the above decomposition contain no pure edges.

Lemma 3.4. *For each positive integer n, there exists an equitably 3-coloured 4-cycle decomposition of $C_{4(n)}$.*

Proof. Let the vertex set of $C_{4(n)}$ be $\mathbb{Z}_n \times \{1, 2, 3, 4\}$, where the second number in each ordered pair denotes the part to which the vertex belongs. Colour all the vertices in two parts of $C_{4(n)}$ with one colour, colour all the vertices in another part with a second colour and colour all the vertices in the remaining part with a third colour. An equitably 3-coloured 4-cycle decomposition of $C_{4(n)}$ is given by $((s, 1), (t, 2), (s, 3), (t, 4))$, where $s, t \in \mathbb{Z}_n$. $\square$

Lemma 3.5. *There exists an equitably 3-coloured 4-cycle decomposition of $K_{2(2)}$.*

Proof. This follows immediately since $K_{2,2}$ is isomorphic to a 4-cycle. $\square$

Lemma 3.6. *There exists an equitably 3-coloured 4-cycle decomposition of $K_{7(4)}$.*

Proof. Let the vertex set of $K_{7(4)}$ be $\bigcup_{i=1}^{7}\{0_i, 1_i, 2_i, 3_i\}$. Colour the vertices j_1, j_2, 0_7 and 1_7 black, colour the vertices j_3, j_4 and 2_7 white and colour the vertices j_5, j_6 and 3_7 grey, where $0 \leqslant j \leqslant 3$. A suitable decomposition is given by

$$
\begin{array}{llllll}
(0_1, 3_2, 1_3, 0_5), & (2_1, 1_2, 3_3, 3_6), & (1_1, 3_2, 1_4, 0_5), & (3_1, 0_2, 0_4, 3_6), & (1_2, 0_1, 3_3, 0_5), & (0_2, 1_1, 1_3, 1_6), \\
(0_2, 2_1, 1_4, 2_5), & (3_2, 3_1, 0_4, 2_6), & (0_1, 0_2, 0_3, 3_7), & (2_1, 2_2, 2_7, 0_5), & (1_1, 2_2, 2_4, 3_7), & (3_1, 1_2, 2_7, 2_6), \\
(2_2, 3_1, 2_3, 3_7), & (2_2, 0_1, 2_7, 1_5), & (3_2, 2_1, 3_4, 3_7), & (1_2, 1_1, 2_7, 3_6), & (0_1, 0_7, 0_3, 1_5), & (1_1, 0_7, 2_3, 2_6), \\
(2_1, 0_7, 2_4, 1_5), & (3_1, 0_7, 3_4, 0_6), & (0_2, 1_7, 1_3, 3_5), & (1_2, 1_7, 0_3, 2_6), & (2_2, 1_7, 3_4, 0_5), & (3_2, 1_7, 2_4, 3_6), \\
(1_7, 0_1, 2_3, 1_5), & (1_7, 1_1, 0_4, 2_5), & (1_7, 2_1, 0_3, 0_6), & (1_7, 3_1, 3_4, 3_6), & (0_7, 0_2, 2_3, 0_5), & (0_7, 1_2, 1_4, 3_5), \\
(0_7, 2_2, 0_3, 1_6), & (0_7, 3_2, 2_4, 2_6), & (2_5, 3_6, 2_3, 1_1), & (0_5, 3_6, 0_3, 3_1), & (2_6, 2_5, 2_3, 2_1), & (3_6, 1_5, 3_3, 1_1), \\
(1_6, 0_5, 2_4, 3_1), & (3_5, 2_6, 1_4, 1_1), & (3_5, 0_6, 0_4, 2_1), & (1_6, 3_5, 2_4, 2_1), & (0_5, 2_6, 3_3, 3_2), & (1_6, 2_5, 1_3, 1_2), \\
(2_5, 0_6, 1_3, 2_2), & (0_6, 0_5, 0_3, 1_2), & (0_6, 1_5, 3_4, 3_2), & (1_5, 1_6, 0_4, 1_2), & (1_5, 2_6, 3_4, 0_2), & (3_6, 3_5, 0_4, 2_2), \\
(3_7, 2_5, 3_3, 2_1), & (3_7, 0_6, 1_4, 3_1), & (3_7, 3_5, 2_3, 1_2), & (3_7, 1_6, 1_4, 0_2), & (1_5, 3_7, 1_3, 0_7), & (0_5, 3_7, 3_3, 1_7), \\
(2_6, 3_7, 0_4, 1_7), & (3_6, 3_7, 1_4, 0_7), & (2_3, 0_4, 0_1, 0_6), & (3_3, 2_4, 1_1, 0_6), & (3_3, 1_4, 0_1, 3_5), & (3_3, 3_4, 1_1, 1_6), \\
(2_4, 0_3, 3_2, 1_6), & (3_4, 1_3, 0_1, 1_6), & (1_4, 1_3, 3_1, 1_5), & (1_4, 0_3, 0_1, 3_6), & (1_3, 2_4, 0_2, 3_6), & (0_3, 3_4, 1_2, 3_5), \\
(2_3, 1_4, 2_2, 1_6), & (1_3, 0_4, 3_2, 1_5), & (0_4, 0_3, 1_1, 1_5), & (2_4, 2_3, 2_2, 0_6), & (3_4, 2_3, 3_2, 3_5), & (0_4, 3_3, 0_2, 0_5), \\
(0_3, 2_7, 2_1, 2_5), & (2_4, 2_7, 3_1, 2_5), & (1_3, 2_7, 0_2, 2_6), & (3_4, 2_7, 3_2, 2_5), & (2_7, 3_3, 0_7, 2_5), & (2_7, 2_3, 1_7, 3_5), \\
(2_7, 0_4, 0_7, 0_6), & (2_7, 1_4, 1_7, 1_6), & (2_1, 1_3, 0_2, 0_6), & (3_1, 3_3, 2_2, 3_5), & (0_1, 2_4, 1_2, 2_5), & (0_1, 3_4, 2_2, 2_6).
\end{array}
$$

$\square$

Lemma 3.7. *There exists an equitably 3-coloured 4-cycle decomposition of $K_{7(6)}$.*

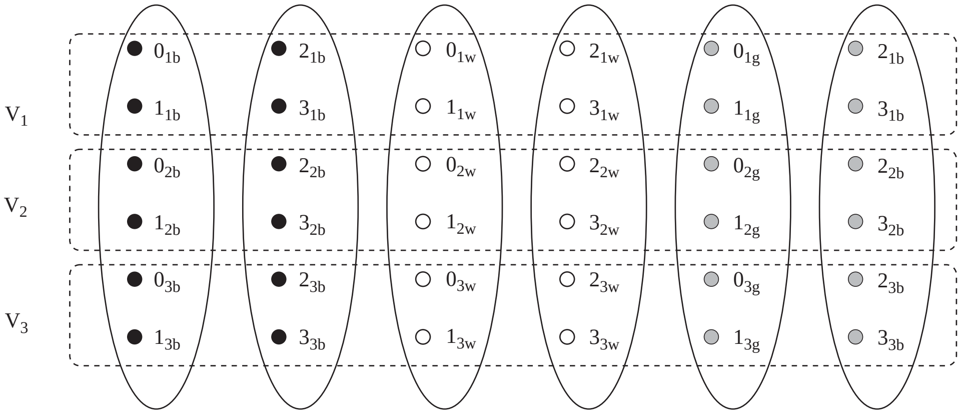

Fig. 1. The arrangement of vertices in graph $G = K_{6(6)}$, with parts grouped vertically.

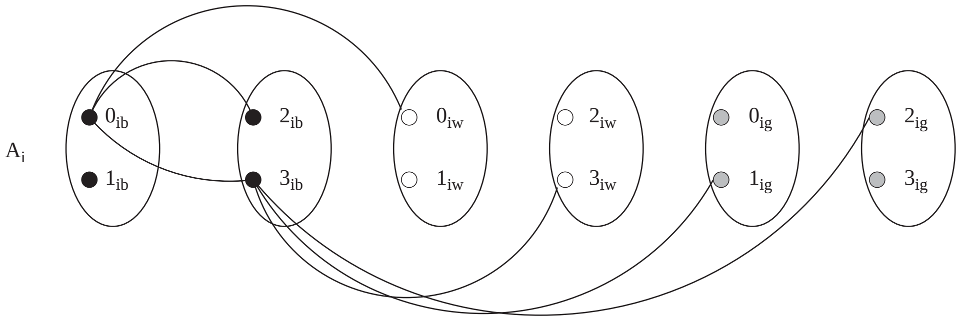

Fig. 2. A partial copy of the graph $A_i = K_{6(2)} = K_{12} - F$, $i \in \{1, 2, 3\}$.

Proof. Within this proof it is important to keep careful note of the colour of each vertex. For convenience, we introduce an additional subscript, using the letters b, w and g to denote the colours black, white and grey, respectively. Thus, for example, the vertex 0_{1w} is coloured white.

Let $G = K_{6(6)}$ on $V_1 \cup V_2 \cup V_3$, where $V_i = \{0_{ix}, 1_{ix}, 2_{ix}, 3_{ix}\}$, for $i = 1, 2, 3$ and $x \in \{b, w, g\}$. Let the parts of G be $\{0_{1x}, 1_{1x}, 0_{2x}, 1_{2x}, 0_{3x}, 1_{3x}\}$ and $\{2_{1x}, 3_{1x}, 2_{2x}, 3_{2x}, 2_{3x}, 3_{3x}\}$, for $x \in \{b, w, g\}$; refer to Fig. 1.

We now decompose G into six subgraphs, A_1, A_2, A_3, B_{23}, B_{13} and B_{12}. Edges with both ends in V_i go into A_i, while edges with one end in V_i and the other in V_j go into B_{ij}, for $1 \leqslant i < j \leqslant 3$. Thus, A_1, A_2 and A_3 are copies of $K_{6(2)}$ (see Fig. 2), while B_{12}, B_{13} and B_{23} are copies of $K_{6(4)} - (2 \times K_{6(2)})$; see Fig. 3.

Note that $K_{6(2)} \cong K_{12} - F$. From Lemma 3.3, we have an equitably 3-coloured 4-cycle decomposition of $K_{12} - F$, containing three cycles without a pure edge, each marked with an asterisk. The first and second of these cycles are of particular interest.

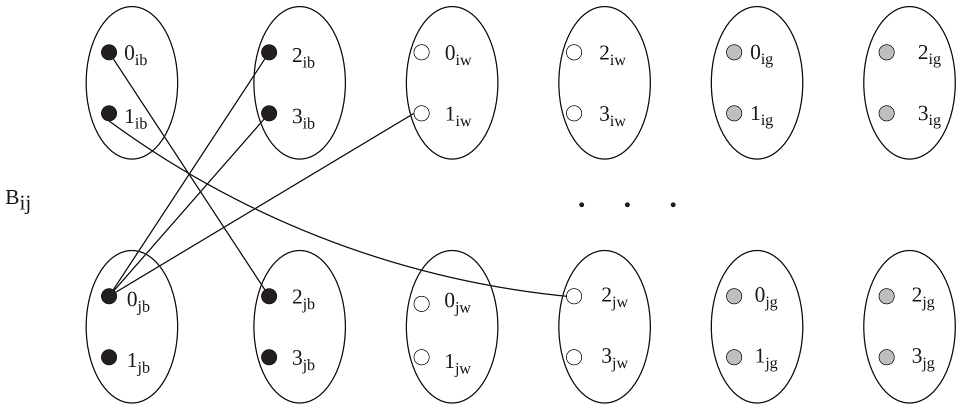

Fig. 3. A partial copy of the graph $B_{ij} = K_{6(4)} - (2 \times K_{6(2)})$, $1 \leqslant i < j \leqslant 3$.

We begin by relabelling the decomposition in Lemma 3.3 so that vertices with subscript 1 now have subscript $1b$, vertices with subscript 2 now have subscript $1w$ and vertices with subscript 3 now have subscript $1g$. Call this decomposition D_1 and place a copy of D_1 on A_1. Note that the two highlighted 4-cycles without a pure edge are relabelled to $(0_{1w}, 1_{1g}, 2_{1b}, 2_{1g})$ and $(1_{1g}, 1_{1w}, 3_{1b}, 2_{1w})$. Then, using the permutation $P = (1b2w3g)(2b3w1g)(3b1w2g)$, place a copy of $P(D_1)$ on A_2 and $P^2(D_1)$ on A_3.

We now consider the graphs B_{ij}, $1 \leqslant i < j \leqslant 3$. We can generate an equitably 3-coloured 4-cycle decomposition of B_{23} based on D_1, our decomposition of A_1. For each 4-cycle $(a_{1p}, c_{1q}, d_{1r}, e_{1s})$ in D_1, where $a, c, d, e \in \{0, 1, 2, 3\}$ and $p, q, r, s \in \{b, w, g\}$, we place two cycles in D_2: $(a_{2p}, c_{3q}, d_{2r}, e_{3s})$ and $(a_{3p}, c_{2q}, d_{3r}, e_{2s})$. D_2 will then be an equitably 3-coloured 4-cycle decomposition of B_{23}. Included in the cycles without a pure edge in D_2 are $(0_{2w}, 1_{3g}, 2_{2b}, 2_{3g})$, $(1_{2g}, 1_{3w}, 3_{2b}, 2_{3w})$, $(0_{3w}, 1_{2g}, 2_{3b}, 2_{2g})$ and $(1_{3g}, 1_{2w}, 3_{3b}, 2_{2w})$. Furthermore, we can place a copy of $P(D_2)$ on B_{13} and a copy of $P^2(D_2)$ on B_{12}.

Hence, $D_1 + P(D_1) + P^2(D_1) + D_2 + P(D_2) + P^2(D_2)$ is an equitably 3-coloured 4-cycle decomposition of G. Within this decomposition, we are interested in a subset of the 4-cycles with no pure edges, given by

$$\begin{array}{llll}
(0_{1w}, 1_{1g}, 2_{1b}, 2_{1g}), & (1_{1g}, 1_{1w}, 3_{1b}, 2_{1w}), & (0_{2g}, 1_{2b}, 2_{2w}, 2_{2b}), & (1_{2b}, 1_{2g}, 3_{2w}, 2_{2g}), \\
(0_{3b}, 1_{3w}, 2_{3g}, 2_{3w}), & (1_{3w}, 1_{3b}, 3_{3g}, 2_{3b}), & (0_{2w}, 1_{3g}, 2_{2b}, 2_{3g}), & (1_{2g}, 1_{3w}, 3_{2b}, 2_{3w}), \\
(0_{3g}, 1_{1b}, 2_{3w}, 2_{1b}), & (1_{3b}, 1_{1g}, 3_{3w}, 2_{1g}), & (0_{1b}, 1_{2w}, 2_{1g}, 2_{2w}), & (1_{1w}, 1_{2b}, 3_{1g}, 2_{2b}), \\
(0_{3w}, 1_{2g}, 2_{3b}, 2_{2g}), & (1_{3g}, 1_{2w}, 3_{3b}, 2_{2w}), & (0_{1g}, 1_{3b}, 2_{1w}, 2_{3b}), & (1_{1b}, 1_{3g}, 3_{1w}, 2_{3g}), \\
(0_{2b}, 1_{1w}, 2_{2g}, 2_{1w}), & (1_{2w}, 1_{1b}, 3_{2g}, 2_{1b}) & & .
\end{array}$$

Let the cycles above be contained in the set R and note that the first and third vertices of these cycles together comprise the set $V_1 \cup V_2 \cup V_3$ with no repetitions.

Let H be $K_{7(6)}$ on $V_1 \cup V_2 \cup V_3 \cup \{b_1, b_2, w_1, w_2, g_1, g_2\}$. The new vertices are in Part 7, with the letters b, w and g denoting the colour of the vertices. To complete the decomposition of $K_{7(6)}$ we must deal with the edges between vertices in Part 7 and all other vertices. This is achieved by removing the cycles in R and adding the following 4-cycles:

$$
\begin{array}{llll}
(0_{1w}, 1_{1g}, 2_{1b}, w_2), & (0_{1w}, b_1, 2_{1b}, 2_{1g}), & (0_{1w}, w_1, 2_{1b}, g_2), & (0_{1w}, g_1, 2_{1b}, b_2),\\
(1_{1g}, 1_{1w}, 3_{1b}, g_1), & (1_{1g}, b_2, 3_{1b}, 2_{1w}), & (1_{1g}, b_1, 3_{1b}, w_2), & (1_{1g}, w_1, 3_{1b}, g_2),\\
(0_{2g}, 1_{2b}, 2_{2w}, w_1), & (0_{2g}, g_2, 2_{2w}, 2_{2b}), & (0_{2g}, b_1, 2_{2w}, w_2), & (0_{2g}, g_1, 2_{2w}, b_2),\\
(1_{2b}, 1_{2g}, 3_{2w}, b_1), & (1_{2b}, w_2, 3_{2w}, 2_{2g}), & (1_{2b}, w_1, 3_{2w}, g_2), & (1_{2b}, g_1, 3_{2w}, b_2),\\
(0_{3b}, 1_{3w}, 2_{3g}, g_1), & (0_{3b}, b_2, 2_{3g}, 2_{3w}), & (0_{3b}, b_1, 2_{3g}, w_2), & (0_{3b}, w_1, 2_{3g}, g_2),\\
(1_{3w}, 1_{3b}, 3_{3g}, w_1), & (1_{3w}, g_2, 3_{3g}, 2_{3b}), & (1_{3w}, b_1, 3_{3g}, w_2), & (1_{3w}, g_1, 3_{3g}, b_2),\\
(0_{2w}, 1_{3g}, 2_{2b}, b_1), & (0_{2w}, w_2, 2_{2b}, 2_{3g}), & (0_{2w}, w_1, 2_{2b}, g_2), & (0_{2w}, g_1, 2_{2b}, b_2),\\
(1_{2g}, 1_{3w}, 3_{2b}, g_1), & (1_{2g}, b_2, 3_{2b}, 2_{3w}), & (1_{2g}, b_1, 3_{2b}, w_2), & (1_{2g}, w_1, 3_{2b}, g_2),\\
(0_{3g}, 1_{1b}, 2_{3w}, w_1), & (0_{3g}, g_2, 2_{3w}, 2_{1b}), & (0_{3g}, b_1, 2_{3w}, w_2), & (0_{3g}, g_1, 2_{3w}, b_2),\\
(1_{3b}, 1_{1g}, 3_{3w}, b_1), & (1_{3b}, w_2, 3_{3w}, 2_{1g}), & (1_{3b}, w_1, 3_{3w}, g_2), & (1_{3b}, g_1, 3_{3w}, b_2),\\
(0_{1b}, 1_{2w}, 2_{1g}, g_1), & (0_{1b}, b_2, 2_{1g}, 2_{2w}), & (0_{1b}, b_1, 2_{1g}, w_2), & (0_{1b}, w_1, 2_{1g}, g_2),\\
(1_{1w}, 1_{2b}, 3_{1g}, w_1), & (1_{1w}, g_2, 3_{1g}, 2_{2b}), & (1_{1w}, b_1, 3_{1g}, w_2), & (1_{1w}, g_1, 3_{1g}, b_2),\\
(0_{3w}, 1_{2g}, 2_{3b}, b_1), & (0_{3w}, w_2, 2_{3b}, 2_{2g}), & (0_{3w}, w_1, 2_{3b}, g_2), & (0_{3w}, g_1, 2_{3b}, b_2),\\
(1_{3g}, 1_{2w}, 3_{3b}, g_1), & (1_{3g}, b_2, 3_{3b}, 2_{2w}), & (1_{3g}, b_1, 3_{3b}, w_2), & (1_{3g}, w_1, 3_{3b}, g_2),\\
(0_{1g}, 1_{3b}, 2_{1w}, w_1), & (0_{1g}, g_2, 2_{1w}, 2_{3b}), & (0_{1g}, b_1, 2_{1w}, w_2), & (0_{1g}, g_1, 2_{1w}, b_2),\\
(1_{1b}, 1_{3g}, 3_{1w}, b_1), & (1_{1b}, w_2, 3_{1w}, 2_{3g}), & (1_{1b}, w_1, 3_{1w}, g_2), & (1_{1b}, g_1, 3_{1w}, b_2),\\
(0_{2b}, 1_{1w}, 2_{2g}, g_1), & (0_{2b}, b_2, 2_{2g}, 2_{1w}), & (0_{2b}, b_1, 2_{2g}, w_2), & (0_{2b}, w_1, 2_{2g}, g_2),\\
(1_{2w}, 1_{1b}, 3_{2g}, w_1), & (1_{2w}, g_2, 3_{2g}, 2_{1b}), & (1_{2w}, b_1, 3_{2g}, w_2), & (1_{2w}, g_1, 3_{2g}, b_2).
\end{array}
$$

Thus, we have an equitably 3-coloured 4-cycle decomposition of $K_{7(6)}$. □

Lemma 3.8. *There exists an equitably* 3-*coloured* 4-*cycle decomposition of* $K_{7(10)} - (K_{7(4)} + K_{7(6)})$.

Proof. Let G be a copy of $K_{6(10)} - (K_{6(6)} + K_{6(4)})$ on $V_1 \cup V_2 \cup V_3 \cup U_1 \cup U_2$, where $V_1 = \bigcup_{i=1}^{6}\{0_i, 1_i\}$, $V_2 = \bigcup_{i=1}^{6}\{2_i, 3_i\}$, $V_3 = \bigcup_{i=1}^{6}\{4_i, 5_i\}$, $U_1 = \bigcup_{i=1}^{6}\{a_i, b_i\}$ and $U_2 = \bigcup_{i=1}^{6}\{c_i, d_i\}$. Let vertices with the same subscript be within the same part. Hence, there is an edge between each vertex in $V_1 \cup V_2 \cup V_3$ and each vertex with a *different* subscript in $U_1 \cup U_2$; see Fig. 4. Colour vertices with subscripts 1 or 2 black, colour vertices with subscripts 3 or 4 white and colour vertices with subscripts 5 or 6 grey.

We now decompose G into six copies of $K_{6(4)} - (2 \times K_{6(2)})$. Let A_{ij} be the subgraph containing the edges between V_i and U_j, for $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 2$; see Fig. 5 for example. We will use the decomposition of $K_{12} - F$ given in Lemma 3.3 to obtain an equitably 3-coloured 4-cycle decomposition of $K_{6(4)} - (2 \times K_{6(2)})$.

We proceed by relabelling the decomposition of $K_{12} - F$ so that we obtain a decomposition on V_1 containing the following three cycles with pure edges:

$$(0_3, 1_5, 0_2, 0_6), (1_5, 1_3, 1_2, 0_4), (1_5, 0_1, 1_4, 1_1).$$

Now let S be a mapping $V_1 \to U_1$ defined by $S(0_x) = a_x$ and $S(1_x) = b_x$. Note that S maps to vertices of like colour. We generate an equitably 3-coloured 4-cycle decomposition of A_{11} as follows: for each 4-cycle (w, x, y, z) in the relabelled $K_{12} - F$ decomposition, we take the 4-cycles

$$(w, S(x), y, S(z)) \text{ and } (S(w), x, S(y), z).$$

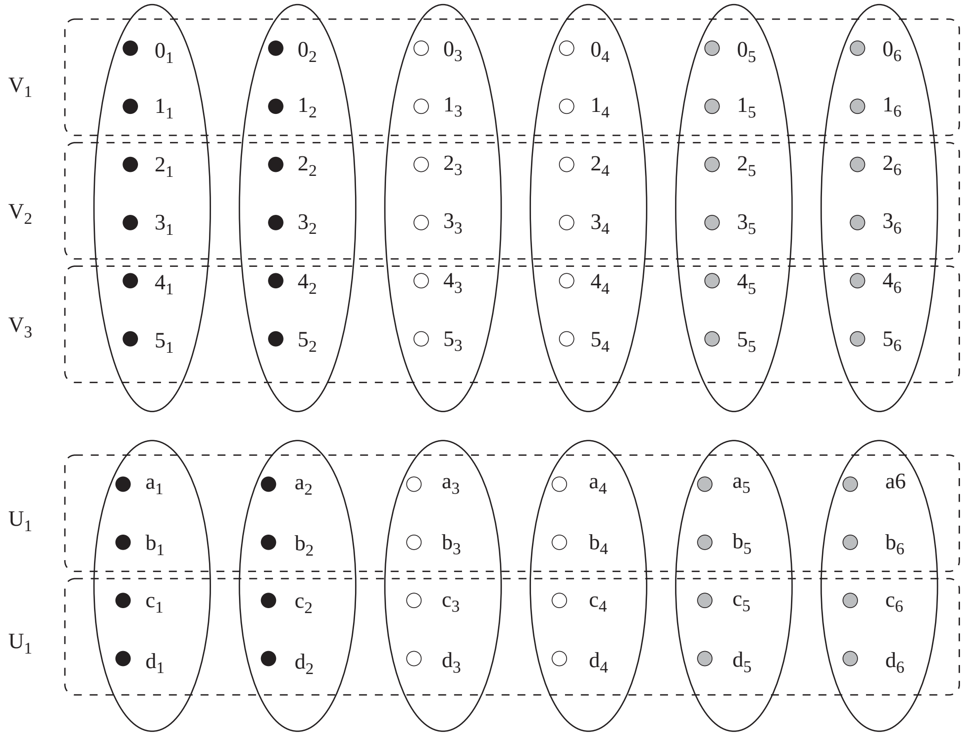

Fig. 4. The graph $G = K_{6(10)} - (K_{6(6)} + K_{6(4)})$, with parts grouped vertically.

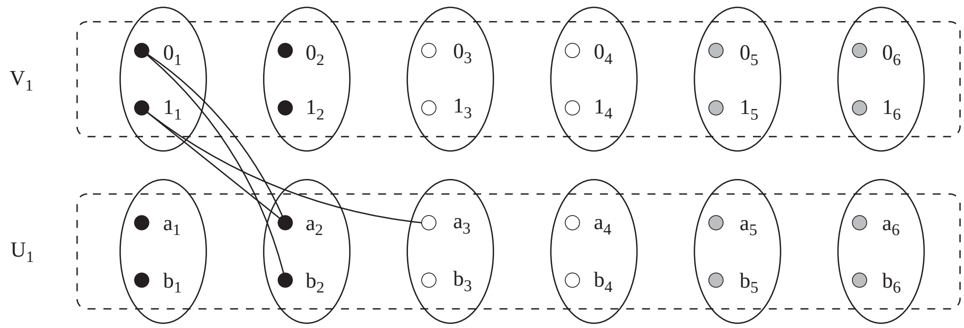

Fig. 5. A partial copy of the graph $A_{11} = K_{6(4)} - (2 \times K_{6(2)})$.

Together these form an equitably 3-coloured 4-cycle decomposition of A_{11}, which we label D_1. Let $D_2 = (1\,2)(3\,5)(4\,6)D_1$ and $D_3' = (1\,3)(2\,4)(56)D_1$, where the permutations are of the part numbers only. Furthermore, let $D_3 = (a_1\,b_1)(b_6\,a_6)D_3'$.

We can now use permutations of D_1, D_2 and D_3 to obtain equitably 3-coloured 4-cycle decompositions of A_{ij}, for $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 2$. Place a copy of D_1 on A_{11}, a copy of

$(a\,c)(b\,d)D_2$ on A_{12}, a copy of $(0\,2)(1\,3)D_2$ on A_{21}, a copy of $(0\,2)(1\,3)(a\,c)(b\,d)D_1$ on A_{22}, a copy of $(0\,4)(1\,5)D_3$ on A_{31} and a copy of $(0\,5)(1\,4)(a\,c)(b\,d)D_3$ on A_{32}. In this case, we are permuting vertices within parts. Thus, an equitably 3-coloured 4-cycle decomposition of G is given by $D_1 + (a\,c)(b\,d)D_2 + (0\,2)(1\,3)D_2 + (0\,2)(1\,3)(a\,c)(b\,d)D_1 + (0\,4)(1\,5)D_3 + (0\,5)(1\,4)(a\,c)(b\,d)D_3$.

This decomposition contains 36 4-cycles without a pure edge. Let the set R contain the following 24 of these cycles:

$$\begin{array}{llllll}
(0_3, b_5, 0_2, a_6), & (1_5, b_3, 1_2, a_4), & (a_3, 1_5, a_2, 0_6), & (b_5, 0_1, b_4, 1_1), & (0_5, d_3, 0_1, c_4), & (1_3, d_5, 1_1, c_6), \\
(c_5, 1_3, c_1, 0_4), & (d_3, 0_2, d_6, 1_2), & (2_5, b_3, 2_1, a_4), & (3_3, b_5, 3_1, a_6), & (a_5, 3_3, a_1, 2_4), & (b_3, 2_2, b_6, 3_2), \\
(2_3, d_5, 2_2, c_6), & (3_5, d_3, 3_2, c_4), & (c_3, 3_5, c_2, 2_6), & (d_5, 2_1, d_4, 3_1), & (4_1, a_6, 4_4, a_5), & (5_6, a_3, 5_2, b_3), \\
(b_1, 5_6, a_4, 4_5), & (a_6, 4_3, b_2, 5_3), & (5_1, c_6, 5_4, c_5), & (4_6, c_3, 4_2, d_3), & (d_1, 4_6, c_4, 5_5), & (c_6, 5_3, d_2, 4_3).
\end{array}$$

Note that the first and third vertices of these cycles together comprise the set $V_1 \cup V_2 \cup V_3 \cup U_1 \cup U_2 - S$ with no repetitions, where $S \subset V_1 \cup V_2 \cup V_3$ consists of six white and six grey vertices.

Let H be $K_{7(10)} - (K_{7(6)} + K_{7(4)})$ on $V_1 \cup V_2 \cup V_3 \cup U_1 \cup U_2 \cup \{0_7, 1_7, \ldots, 5_7, a_7, b_7, c_7, d_7\}$. Colour the vertices $0_7, 2_7, a_7$ and c_7 black, colour the vertices $1_7, 4_7$ and b_7 white and colour the vertices 3_7, 5_7 and d_7 grey.

To complete the decomposition we must consider the edges between $V_1 \cup V_2 \cup V_3$ and $\{a_7, b_7, c_7, d_7\}$ and those between $U_1 \cup U_2$ and $\{0_7, 1_7, \ldots, 5_7\}$. This is done by removing the 4-cycles in R and adding the following 4-cycles:

$$\begin{array}{llllll}
(0_3, b_5, 0_2, a_7), & (0_3, b_7, 0_2, a_6), & (0_3, c_7, 0_2, d_7), & (1_5, b_3, 1_2, c_7), & (1_5, d_7, 1_2, a_4), & (1_5, a_7, 1_2, b_7), \\
(0_5, d_3, 0_1, c_7), & (0_5, d_7, 0_1, c_4), & (0_5, a_7, 0_1, b_7), & (1_3, d_5, 1_1, a_7), & (1_3, b_7, 1_1, c_6), & (1_3, c_7, 1_1, d_7), \\
(2_5, b_3, 2_1, c_7), & (2_5, d_7, 2_1, a_4), & (2_5, a_7, 2_1, b_7), & (3_3, b_5, 3_1, a_7), & (3_3, b_7, 3_1, a_6), & (3_3, c_7, 3_1, d_7), \\
(2_3, d_5, 2_2, a_7), & (2_3, b_7, 2_2, c_6), & (2_3, c_7, 2_2, d_7), & (3_5, d_3, 3_2, c_7), & (3_5, d_7, 3_2, c_4), & (3_5, a_7, 3_2, b_7), \\
(4_1, a_6, 4_4, a_7), & (4_1, b_7, 4_4, a_5), & (4_1, c_7, 4_4, d_7), & (5_6, a_3, 5_2, c_7), & (5_6, d_7, 5_2, b_3), & (5_6, a_7, 5_2, b_7), \\
(5_1, c_6, 5_4, a_7), & (5_1, b_7, 5_4, c_5), & (5_1, c_7, 5_4, d_7), & (4_6, c_3, 4_2, c_7), & (4_6, d_7, 4_2, d_3), & (4_6, a_7, 4_2, b_7), \\
(0_4, a_7, 0_6, b_7), & (0_4, c_7, 0_6, d_7), & (1_4, a_7, 1_6, b_7), & (1_4, c_7, 1_6, d_7), & (2_4, a_7, 2_6, b_7), & (2_4, c_7, 2_6, d_7), \\
(3_4, a_7, 3_6, b_7), & (3_4, c_7, 3_6, d_7), & (4_3, a_7, 4_5, b_7), & (4_3, c_7, 4_5, d_7), & (5_3, a_7, 5_5, b_7), & (5_3, c_7, 5_5, d_7), \\
(a_3, 1_5, a_2, 0_7), & (a_3, 1_7, a_2, 0_6), & (a_3, 2_7, a_2, 3_7), & (a_3, 4_7, a_2, 5_7), & (b_5, 0_1, b_4, 4_7), & (b_5, 5_7, b_4, 1_1), \\
(b_5, 0_7, b_4, 1_7), & (b_5, 2_7, b_4, 3_7), & (c_5, 1_3, c_1, 2_7), & (c_5, 3_7, c_1, 0_4), & (c_5, 0_7, c_1, 1_7), & (c_5, 4_7, c_1, 5_7), \\
(d_3, 0_2, d_6, 4_7), & (d_3, 5_7, d_6, 1_2), & (d_3, 0_7, d_6, 1_7), & (d_3, 2_7, d_6, 3_7), & (a_5, 3_3, a_1, 2_7), & (a_5, 3_7, a_1, 2_4), \\
(a_5, 0_7, a_1, 1_7), & (a_5, 4_7, a_1, 5_7), & (b_3, 2_2, b_6, 4_7), & (b_3, 5_7, b_6, 3_2), & (b_3, 0_7, b_6, 1_7), & (b_3, 2_7, b_6, 3_7), \\
(c_3, 3_5, c_2, 0_7), & (c_3, 1_7, c_2, 2_6), & (c_3, 2_7, c_2, 3_7), & (c_3, 4_7, c_2, 5_7), & (d_5, 2_1, d_4, 4_7), & (d_5, 5_7, d_4, 3_1), \\
(d_5, 0_7, d_4, 1_7), & (d_5, 2_7, d_4, 3_7), & (b_1, 5_6, a_4, 0_7), & (b_1, 1_7, a_4, 4_5), & (b_1, 2_7, a_4, 3_7), & (b_1, 4_7, a_4, 5_7), \\
(a_6, 4_3, b_2, 2_7), & (a_6, 3_7, b_2, 5_3), & (a_6, 0_7, b_2, 1_7), & (a_6, 4_7, b_2, 5_7), & (d_1, 4_6, c_4, 0_7), & (d_1, 1_7, c_4, 5_5), \\
(d_1, 2_7, c_4, 3_7), & (d_1, 4_7, c_4, 5_7), & (c_6, 5_3, d_2, 2_7), & (c_6, 3_7, d_2, 4_3), & (c_6, 0_7, d_2, 1_7), & (c_6, 4_7, d_2, 5_7).
\end{array}$$

$\square$

Lemma 3.9. *There is no equitably* 3*-colourable* 4*-cycle decomposition of* $K_{8(n)}$.

Proof. Suppose that there are b black, w white and g grey vertices in $K_{8(n)}$ such that $b + w + g = 8n$. Without loss of generality let $b \leqslant w \leqslant g$. We now consider edges between a black vertex and a vertex of a different colour. We will call edges of this type $b*$ edges. Note that each equitably 3-coloured 4-cycle contains at least two $b*$ edges and that a 4-cycle decomposition of $K_{8(n)}$ contains $7n^2$ 4-cycles. Thus, an equitably 3-colourable 4-cycle decomposition is possible only if $K_{8(n)}$ contains at least $14n^2$ $b*$ edges.

The number of $b*$ edges is maximised when $k = \lfloor b/n \rfloor$ parts each contain n black vertices, one part has $d = b - kn$ black vertices and the other $p - k - 1$ parts contain no black vertices.

To prove this, assume that the b black vertices are not arranged as above, but that the number of $b*$ edges is maximised. We now seek a contradiction.

There must be two parts each with some black vertices and some non-black vertices. Say that one part has r black vertices and another has q black vertices, for $0 < r \leqslant q < n$. There are $r(n-q)+q(n-r)b*$ edges between these two parts. We can interchange the colours of a black vertex from the first part with a non-black vertex from the other part, without changing the total number of black vertices or the number of $b*$ edges between these two parts and the other $p-2$ parts.

These two parts now have $r-1$ and $q+1$ black vertices, respectively, and there are $(r-1)(n-q-1)+(q+1)(n-r+1)b*$ edges between them. However, $[(r-1)(n-q-1)+(q+1)(n-r+1)]-[r(n-q)+q(n-r)]=2q-2r+2>0$ since $q \geqslant r$, which is a contradiction.

Using the configuration which maximises the number of $b*$ edges, there are $n^2(k^2+k)+nb(7-2k)$ such edges, which is a strictly increasing function of b provided $k \leqslant 3$. But $b \leqslant w \leqslant g$, so $b \leqslant 8n/3$ and $k \leqslant \lfloor 8/3 \rfloor = 2$.

Thus, the number of $b*$ edges is maximised when $b=8n/3$ (giving $k=2$), and to produce this maximum we need to have two parts containing all black vertices and one other part with two-thirds of its vertices coloured black.

The number of $b*$ edges is then $14n^2$. Since this is also the minimum number of $b*$ edges, we must use the configuration from the preceding paragraph.

If we now consider white vertices, we see that in order for there to be at least $14n^2$ edges between white vertices and vertices of another colour, the white vertices need to be arranged in the same manner as the black vertices. Hence we are forced to arrange the remaining $v/3$ grey vertices such that two parts each contain n grey vertices and two other parts contain $n/3$ grey vertices.

However, we find that there are $40n^2/3 < 14n^2$ edges connecting a grey vertex and a vertex of a different colour. □

Theorem 3.10. *There exist equitably* 3-*colourable* 4-*cycle decompositions of* $K_{p(n)}$ *if and only if* $p \in \{1,2,3,4,5,6,7,9\}$, $n \equiv 0 \pmod 2$ *and, if* $p=7$, *then* $n \geqslant 4$.

Proof. Consider $K_{p(n)}$ as the complete graph on pn vertices with the edges of p disjoint copies of the complete graph on n vertices removed, denoted $K_{pn}-(p \times K_n)$. This graph has $\frac{1}{2}p(p-1)n^2$ edges.

Suppose that there are b black, w white and g grey vertices in $K_{pn}-(p \times K_n)$. We wish to determine what values of b, w and g maximise the number of mixed edges in the graph. This is done by maximising the number of mixed edges in K_{pn} and removing only pure edges when removing the edges of each disjoint copy of K_n. Using simple calculus we find that when $b=w=g=pn/3$ the number of mixed edges in $K_{p(n)}$ is maximised. Indeed, the maximum possible number of mixed edges in $K_{p(n)}$ is $(pn)^2/3$.

Since a 4-cycle contains at most one pure edge, the number of mixed edges in $K_{p(n)}$ must be at least $\frac{3}{4}$ of the total number of edges. Hence, $(pn)^2/3 \geqslant 3p(p-1)n^2/8$. Solving this inequality, we find that $p \leqslant 9$. Combining this with Lemma 3.9, we find that an equitably 3-colourable 4-cycle decomposition of $K_{p(n)}$ can exist only if $p \in \{1,2,3,4,5,6,7,9\}$.

Furthermore, for each permitted value of p, when $n \equiv 0 \pmod 2$ the number of edges in $K_{p(n)}$ is divisible by 4, each vertex has even degree and $pn \geqslant 4$.

We now provide equitably 3-coloured 4-cycle decompositions of $K_{p(n)}$ for $p \in \{1, 2, 3, 4, 5, 6, 9\}$, whenever $n \equiv 0 \pmod 2$, $n \geqslant 2$, and an equitably 3-coloured 4-cycle decomposition of $K_{7(n)}$ whenever $n \equiv 0 \pmod 2$, $n \geqslant 4$.

Case 1: $p = 1$.

Since $K_{1(n)}$ has no edges, an equitably 3-coloured 4-cycle decomposition of $K_{1(n)}$ exists trivially.

Case 2: $p = 2$.

Let $n = 2x$. Take a copy of $K_{2(x)}$. We simultaneously construct $K_{2(n)}$ and its decomposition as follows. Replace each vertex of $K_{2(x)}$ with two vertices, in the first part colouring one black and one white and in the second part colouring one black and one grey. By Lemma 3.5, we can place an equitably 3-coloured 4-cycle decomposition of $K_{2(2)}$ on each set of vertices arising from an edge of $K_{2(x)}$. The result is an equitably 3-coloured 4-cycle decomposition of $K_{2(n)}$.

Case 3: $p = 3$.

As with Case 2, let $n = 2x$. Take a copy of $K_{3(x)}$ and replace each vertex with two vertices, in the first part colouring one black and one white, in the second part colouring one black and one grey and in the third part colouring one white and one grey. We now have a copy of $K_{3(n)}$. By Lemma 3.5, we can place an equitably 3-coloured 4-cycle decomposition of $K_{2(2)}$ on each set of vertices arising from an edge of $K_{3(x)}$. The result is an equitably 3-coloured 4-cycle decomposition of $K_{3(n)}$.

Case 4: $p = 4$.

Let $n = 2x$. Take a copy of $K_8 - F \cong K_{4(2)}$. By Lemma 3.2, we can place an equitably 3-coloured 4-cycle decomposition on this graph. Now replace each vertex of $K_8 - F$ by x new vertices, colouring the new vertices the same colour as the vertex they replaced. Thus, we have a copy of $K_{4(n)}$ and each 4-cycle in the original decomposition becomes a copy of $C_{4(x)}$, resulting in a $C_{4(x)}$ decomposition of $K_{4(n)}$. Without loss of generality, all vertices in parts 1 and 2 of each copy of $C_{4(x)}$ are of one colour, all vertices in part 3 are of a second colour and all vertices in part 4 are of a third colour. By Lemma 3.4, we can place an equitably 3-coloured 4-cycle decomposition of $C_{4(x)}$ on each copy of $C_{4(x)}$ in the $C_{4(x)}$ decomposition of $K_{4(n)}$. The result is an equitably 3-coloured 4-cycle decomposition of $K_{4(n)}$.

Case 5: $p = 5$.

Use Lemma 3.2 and exactly the same procedure as for Case 4, but replace the graph $K_8 - F$ with $K_{10} - F \cong K_{5(2)}$.

Case 6: $p = 6$.

Use Lemma 3.2 and exactly the same procedure as for Case 4, but replace the graph $K_8 - F$ with $K_{12} - F \cong K_{6(2)}$.

Case 7: $p = 7$.

By Lemma 3.2, no equitably 3-coloured 4-cycle decomposition of $K_{7(2)}$ exists. Hence, let $n \geqslant 4$. We need to consider two cases.

Case 7.1: $n \equiv 0 \pmod 4$.

Let $n = 4x$ and take a copy of $K_{7(4)}$. By Lemma 3.6, there exists an equitably 3-coloured 4-cycle decomposition of $K_{7(4)}$. We simultaneously create the graph $K_{7(n)}$, $n \geqslant 4$, and its equitably 3-coloured 4-cycle decomposition as follows. Replace each vertex by x vertices, colouring them the same colour as the vertex they replaced. By Lemma 3.4, we can place an equitably 3-coloured 4-cycle decomposition of $C_{4(n)}$ on each set of vertices arising from a 4-cycle in the decomposition of $K_{7(4)}$. The result is an equitably 3-coloured 4-cycle decomposition of $K_{7(n)}$, where $n \equiv 0 \pmod 4$.

Case 7.2: $n \equiv 2 \pmod 4$.

Let $n = 4x + 2$. An equitably 3-coloured 4-cycle decomposition of $K_{7(6)}$, (so $x = 1$) is given in Lemma 3.7. Suppose then that $x \geqslant 2$. Take a copy of $K_{7(6)}$, with vertex set V. Call this graph G_1 Colour the vertices in parts 1 and 2 black. Colour the vertices in parts 3 and 4 white and colour the vertices in parts 5 and 6 grey. Finally, colour two vertices each black, white and grey in part 7. Now take a copy of $K_{7(4(x-1))}$. Call this graph G_2. Colour the vertices in parts 1–6 as above and colour $2(x - 1)$ vertices in part 7 black and $x - 1$ each white and grey.

Partition the vertices of $K_{7(4(x-1))}$ into $x - 1$ sets, $U_1, U_2, \ldots, U_{x-1}$, such that each set contains four vertices from each part, including two black, one white and one grey from Part 7.

For $1 \leqslant i \leqslant x - 1$, include the set of edges between V and U_i, with the exception of edges between vertices in the same part. Let the graph with vertex set $V \cup U_i$ and edge set as described above be called H_i, for $1 \leqslant i \leqslant x - 1$. Then $H_i \cong K_{7(10)} - (K_{7(6)} + K_{7(4)})$, for $1 \leqslant i \leqslant x - 1$, with the colouring pattern specified in Lemma 3.8.

The union of G_1, G_2 and $H_1, \ldots, H_{x-1}$ is the graph $K_{7(n)}$. By Lemmas 3.7 and Case 7.1 of this theorem, we can place an equitably 3-coloured 4-cycle decomposition of $K_{7(6)}$ and $K_{7(4(x-1))}$ on G_1 and G_2, respectively. Similarly, by Lemma 3.8, we can place an equitably 3-coloured 4-cycle decomposition of $K_{7(10)} - (K_{7(6)} + K_{7(4)})$ on H_i, for $1 \leqslant i \leqslant x - 1$. Thus, we have an equitably 3-coloured 4-cycle decomposition of $K_{7(n)}$.

Case 8: $p = 9$.

Take a copy of K_9. By Lemma 3.1, an equitably 3-coloured 4-cycle decomposition of K_9 exists. Replace each vertex in the equitably 3-coloured 4-cycle decomposition of K_9 by n vertices, colouring the vertices the same colour as the vertex they replaced. Thus, we have a copy of $K_{9(n)}$. By Lemma 3.4, we can place an equitably 3-coloured 4-cycle decomposition of $C_{4(x)}$ on each set of vertices arising from a 4-cycle in the 4-cycle decomposition of K_9. The result is an equitably 3-coloured 4-cycle decomposition of $K_{9(n)}$. $\square$

4. Equitably 3-colourable 5-cycle decompositions

Lemma 4.1. *For each positive integer n, there exists an equitably 3-coloured 5-cycle decomposition of $C_{5(n)}$.*

Proof. Let the vertex set of $C_{5(n)}$ be $\mathbb{Z}_n \times \{1, 2, \ldots, 5\}$, where the second number in each ordered pair denotes the part to which the vertex belongs. Colour all the vertices in two parts of $C_{5(n)}$ with one colour, all the vertices in another two parts with a second colour and all the vertices in the remaining part with a third colour. An equitably 3-coloured 5-cycle decomposition of $C_{5(n)}$ is given by the starter cycle $((s, 1), (t, 2), (s, 3), (t, 4), (s + t, 5))$, where $s, t \in \mathbb{Z}_n$ and addition is calculated modulo n. $\square$

Lemma 4.2. *If there exists an equitably* 3-*coloured* 5-*cycle decomposition of* $K_{p(n)}$, *then there exists an equitably* 3-*coloured* 5-*cycle decomposition of* $K_{p(nx)}$ *for any positive integer* x.

Proof. Take an equitably 3-coloured 5-cycle decomposition of $K_{p(n)}$. Replace each vertex in the 5-cycle decomposition of $K_{p(n)}$ by x vertices, colouring them the same colour as the vertex they replaced. By Lemma 4.1, we can place an equitably 3-coloured 5-cycle decomposition of $C_{5(x)}$ on each set of vertices arising from a 5-cycle in the decomposition of $K_{p(n)}$. It is easy to check that the result is an equitably 3-coloured 5-cycle decomposition of $K_{p(nx)}$. □

From Lemmas 4.3 and 4.4 below, we can conclude that equitably 3-colourable 5-cycle decompositions of $K_{p(1)}$ and $K_{p(2)}$ exist whenever the obvious necessary conditions are satisfied.

Lemma 4.3 (*Adams et al. [1]*). *There exist equitably* 3-*colourable* 5-*cycle decompositions of* K_v *if and only if* $v \equiv 1, 5 \pmod{10}$, $v \geqslant 5$.

Lemma 4.4 (*Adams et al. [1]*). *There exist equitably* 3-*colourable* 5-*cycle decompositions of* $K_v - F$ *if and only if* $v \equiv 0, 2 \pmod{10}$, $v \geqslant 10$.

Definition (*Colbourn and Rosa [11]*). Suppose that the elements of a PBD$(v, 3, 1)$ or a GDD$[3, 1, M; v]$ have been coloured using k colours. The colouring is said to be *weak* if no block is monochromatic. Blocks with two elements of the same colour are said to be *bicoloured* while blocks with three differently coloured elements are said to be *polychromatic*. The design is said to be *weakly k-colourable* if it permits a weak k-colouring. Furthermore, the *weak chromatic number*, χ_w, is the smallest value of k for which there exists a weak k-colouring, and the design is said to be *weakly k-chromatic* if $\chi_w = k$.

We now consider decompositions of $K_{p(5)}$ and $K_{p(10)}$ as these are needed in the proof of Theorem 4.13.

Lemma 4.5 (*Colbourn and Rosa [11]*). *For all* $v \equiv 1, 3 \pmod 6$, $v \geqslant 3$, *there exists a weakly* 3-*chromatic* PBD$(v, 3, 1)$.

Corollary 4.6. *For all* $v \equiv 0, 2 \pmod 6$, $v \geqslant 6$, *there exists a weakly* 3-*chromatic* GDD$[3, 1, 2; v]$.

Proof. Let $v \equiv 0, 2 \pmod 6$. By Lemma 4.5, there exists a weakly 3-chromatic PBD$(v + 1, 3, 1)$, from which we can remove an element e. The result is a GDD$[3, 1, 2; v]$ where the original blocks not containing e are bicoloured or polychromatic blocks in the design, and the original blocks containing e are now groups within the design. □

Lemma 4.7. *For all* $v \equiv 5 \pmod 6$, *there exists a* PBD$(v, \{3, 5^*\}, 1)$, *whose elements have been coloured using three colours, such that no block of size* 3 *is monochromatic and the block of size* 5 *has one element of one colour and two elements each of the other two colours.*

Proof. Let $v = 6x + 5$. Furthermore, let the elements of the PBD$(v, \{3, 5^*\}, 1)$ be denoted $\{\infty_1, \infty_2\} \cup (\{1, 2, \ldots, 2x + 1\} \times \{1, 2, 3\})$. Using the $6x + 5$ construction given in [14], where the block of size five is on $\{\infty_1, \infty_2, (1, 1), (1, 2), (1, 3)\}$, and colouring the elements ∞_1, $(1, 1), (2, 1), \ldots, (2x + 1, 1)$ with one colour, the elements ∞_2, $(1, 2), (2, 2), \ldots, (2x+1, 2)$ with a second colour and the elements $(1, 3), (2, 3), \ldots, (2x+1, 3)$ with a third colour, we obtain a PBD$(v, \{3, 5^*\}, 1)$ with the required block colouring. □

Corollary 4.8. *For all* $v \equiv 4 \pmod 6$, $v \geqslant 10$, *there exists a* GDD$[\{3, 5^*\}, 1, 2; v]$, *whose elements have been coloured using three colours, such that no block of size* 3 *is monochromatic and the block of size* 5 *has one element of one colour and two elements each of the other two colours.*

Proof. Let $v \equiv 4 \pmod 6$, $v \geqslant 10$. By Lemma 4.5,4.13, there exists a PBD$(v+1, \{3, 5^*\}, 1)$, whose elements have been coloured using three colours, such that no block of size 3 is monochromatic and the block of size 5 has one element of one colour and two elements each of the other two colours. Removing an element e from this design which does not occur in the block of size 5, we obtain a GDD$[\{3, 5^*\}, 1, 2; v]$, where the original blocks not containing e are blocks in the design, and the original blocks containing e are now groups within the design. It is also easy to check that each block has the required colouring pattern. □

Lemma 4.9. *There exist equitably* 3*-coloured* 5*-cycle decompositions of* $K_{3(5)}$.

Proof. We give two such decompositions, denoted *Type A* and *Type B*. The assignment of colours to vertices varies between the two decompositions. In each case, the vertex set is denoted $\bigcup_{i=1}^{3}\{0_i, 1_i, \ldots, 4_i\}$.

Type A. Colour the vertices such that $0_1, 0_2, 0_3$ and 1_3 are one colour, $1_1, 2_1, 1_2, 2_2$ and 2_3 are a second colour and $3_1, 4_1, 3_2, 4_2, 3_3$ and 4_3 are a third colour. An equitably 3-coloured 5-cycle decomposition of this graph is given by

$(1_3, 2_1, 4_3, 2_2, 4_1)$, $(1_3, 2_2, 3_3, 2_1, 3_2)$, $(1_3, 0_1, 4_3, 4_1, 1_2)$, $(1_3, 0_2, 4_3, 3_2, 1_1)$, $(2_1, 1_2, 3_3, 4_1, 0_3)$,
$(2_1, 2_3, 4_2, 4_1, 0_2)$, $(2_2, 1_1, 4_3, 4_2, 0_3)$, $(2_2, 2_3, 3_1, 4_2, 0_1)$, $(2_1, 2_2, 3_1, 1_3, 4_2)$, $(0_1, 0_2, 3_1, 3_2, 2_3)$,
$(0_2, 0_3, 3_2, 3_3, 1_1)$, $(0_3, 0_1, 3_3, 3_1, 1_2)$, $(1_1, 1_2, 4_3, 3_1, 0_3)$, $(1_2, 2_3, 4_1, 3_2, 0_1)$, $(2_3, 1_1, 4_2, 3_3, 0_2)$.

Type B. Colour the vertices such that $0_1, 0_2, 1_2, 0_3$ and 1_3 are one colour, $1_1, 2_1, 2_2, 2_3$ and 3_3 are a second colour and $3_1, 4_1, 3_2, 4_2$ and 4_3 are a third colour. An equitably 3-coloured 5-cycle decomposition is given by

$(0_2, 0_3, 1_1, 2_2, 4_3)$, $(0_3, 1_2, 2_3, 2_2, 4_1)$, $(0_1, 0_2, 2_3, 1_1, 3_2)$, $(0_1, 0_3, 2_1, 3_3, 4_2)$, $(0_2, 1_3, 3_2, 4_3, 1_1)$,
$(1_3, 1_2, 4_1, 4_3, 2_1)$, $(0_1, 1_2, 3_1, 3_2, 2_3)$, $(0_1, 1_3, 4_2, 3_1, 2_2)$, $(3_3, 1_1, 4_2, 4_3, 0_1)$, $(2_1, 2_3, 3_1, 4_3, 1_2)$,
$(2_2, 3_3, 4_1, 4_2, 0_3)$, $(2_2, 2_1, 3_2, 4_1, 1_3)$, $(1_3, 1_1, 1_2, 3_3, 3_1)$, $(0_2, 3_1, 0_3, 3_2, 3_3)$, $(2_1, 4_2, 2_3, 4_1, 0_2)$. □

Lemma 4.10. *There exists an equitably* 3*-coloured* 5*-cycle decomposition of* $K_{5(5)}$.

Proof. Let the vertex set of $K_{5(5)}$ be $\bigcup_{i=1}^{5}\{0_i, 1_i, \ldots, 4_i\}$. Colour the vertices such that 0_1, $1_1, 0_2, 1_2, 0_3, 1_3, 0_4$ and 0_5 are one colour, $2_1, 3_1, 2_2, 2_3, 1_4, 2_4, 1_5$ and 2_5 are a second colour, and 4_1, 3_2, 4_2, 3_3, 4_3, 3_4, 4_4, 3_5 and 4_5 are a third colour. An equitably 3-coloured 5-cycle decomposition of $K_{5(5)}$ is given by

$$\begin{array}{lllll}
(0_1,1_2,2_3,3_4,3_5), & (1_1,0_2,2_3,4_4,4_5), & (0_2,0_3,2_4,3_5,4_1), & (1_2,1_3,1_4,4_5,4_1), & (1_4,2_5,0_1,4_2,3_3),\\
(2_4,1_5,1_1,3_2,4_3), & (1_5,3_1,1_2,4_3,4_4), & (2_5,2_1,0_2,4_3,3_4), & (2_3,1_4,0_5,0_1,3_2), & (2_3,2_4,0_5,1_1,4_2),\\
(4_1,2_2,2_3,0_4,0_5), & (0_1,0_2,3_3,3_4,1_5), & (1_1,1_2,3_3,4_4,2_5), & (0_2,1_3,4_4,3_5,3_1), & (1_2,0_3,3_4,4_5,2_1),\\
(1_4,1_5,4_1,3_2,0_3), & (2_4,2_5,4_1,4_2,1_3), & (1_5,2_1,3_2,3_3,0_4), & (2_5,3_1,4_2,4_3,0_4), & (2_1,2_2,0_3,0_4,3_5),\\
(3_1,2_2,1_3,0_4,4_5), & (0_1,2_2,3_3,2_4,4_5), & (1_1,2_2,4_3,1_4,3_5), & (2_1,4_2,0_3,4_4,0_5), & (3_1,3_2,1_3,3_4,0_5),\\
(1_4,4_1,3_3,0_5,0_2), & (2_4,4_1,4_3,0_5,1_2), & (1_1,3_3,3_5,2_2,1_4), & (0_1,4_3,4_5,2_2,2_4), & (0_3,4_5,3_2,2_4,1_1),\\
(1_3,3_5,4_2,1_4,0_1), & (2_5,3_2,4_4,0_1,0_3), & (1_5,4_2,3_4,1_1,1_3), & (0_2,0_4,2_1,2_3,3_5), & (1_2,0_4,3_1,2_3,4_5),\\
(1_5,0_2,3_4,0_1,3_3), & (2_5,1_2,4_4,1_1,4_3), & (4_1,2_3,0_5,2_2,0_4), & (2_1,4_3,3_5,1_2,1_4), & (3_1,3_3,4_5,0_2,2_4),\\
(0_3,3_5,3_2,1_4,3_1), & (1_3,4_5,4_2,2_4,2_1), & (0_5,3_2,3_4,2_1,0_3), & (0_5,4_2,4_4,3_1,1_3), & (2_2,3_4,4_1,0_3,1_5),\\
(2_2,4_4,4_1,1_3,2_5), & (0_4,0_1,2_3,1_5,3_2), & (0_4,1_1,2_3,2_5,4_2), & (1_2,3_4,3_1,4_3,1_5), & (0_2,4_4,2_1,3_3,2_5).
\end{array}$$

□

We are now in a position to find equitably 3-coloured 5-cycle decompositions of $K_{p(5)}$ and $K_{p(10)}$.

Lemma 4.11. *There exist equitably* 3*-colourable* 5*-cycle decompositions of* $K_{p(5)}$ *if and only if* p *is an odd, positive integer.*

Proof. For such a decomposition to exist, the number of edges must be divisible by five and each vertex must have even degree. The number of edges in $K_{p(5)}$ is always divisible by 5. As each part contains an odd number of vertices, we require there to be an odd number of parts in order to satisfy the second condition. Thus, p is an odd number.

Let p be any odd, positive integer. By Lemmas 4.5 and 4.5,4.13, there exists a PBD$(p, 3, 1)$ or a PBD$(p, \{3, 5^*\}, 1)$ whose vertices have been coloured using three colours, such that no block of size 3 is monochromatic and the one block of size 5 has one vertex of Colour 1 and two each of the other two colours. Replace each vertex of the design by five vertices. If the vertex they replaced was of Colour 1, then colour one vertex black and two each white and grey. If the vertex they replaced was of Colour 2, then colour one vertex white and two each black and grey. Finally, if the vertex they replaced was of Colour 3, then colour one vertex grey and two each black and white. By Lemma 4.9, we can place an equitably 3-coloured 5-cycle decomposition of $K_{3(5)}$ of Type A or B on each set of vertices arising from a bicoloured or polychromatic block of size 3, respectively. Furthermore, by Lemma 4.10, we can place an equitably 3-coloured 5-cycle decomposition of $K_{5(5)}$ on the block of size 5 arising from the design. The result is an equitably 3-coloured 5-cycle decomposition of $K_{p(5)}$. □

Lemma 4.12. *There exist equitably* 3*-colourable* 5*-cycle decompositions of* $K_{p(10)}$ *if and only if* p *is a positive integer such that* $p \neq 2$.

Proof. Once again, the number of edges in the graph must be divisible by five and each vertex must have even degree. The number of edges in $K_{p(10)}$ is always divisible by 5. As each part contains an even number of vertices, the second condition is satisfied for any p. However, no odd-length cycle decomposition of a bipartite graph exists. Hence, p is a positive integer such that $p \neq 2$.

Suppose $p \neq 2$. By Lemmas 4.6 and 4.6,4.13, there exists a GDD$[3, 1, 2, 2p]$ or a GDD$[\{3, 5*\}, 1, 2, 2p]$ for all $p \neq 2$, whose elements have been coloured using three colours, such that no block of size 3 is monochromatic and the block of size 5 has one element of one colour and two elements each of the other two colours. Again, replace each element in the design by 5 elements and colour the new elements according to the method used in the proof of Lemma 4.11. By Lemma 4.9, we can place an equitably 3-coloured 5-cycle decomposition of $K_{3(5)}$ of Type A or B on each set of vertices arising from a bicoloured or polychromatic block of size 3, respectively. Furthermore, by Lemma 4.10, we can place an equitably 3-coloured 5-cycle decomposition of $K_{5(5)}$ on the block of size 5 arising from the design. The result is an equitably 3-coloured 5-cycle decomposition of $K_{p(10)}$. □

We prove the following.

Theorem 4.13. *There exists an equitably* 3-*colourable* 5-*cycle decomposition of* $K_{p(n)}$ *if and only if the number of edges is divisible by* 5, *each vertex has even degree,* $p \neq 2$, *and if* $p \in \{3, 4\}$ then $n > 1$.

Proof. It is shown in [3] that an uncoloured 5-cycle decomposition of $K_{p(n)}$ exists if and only if the conditions given above are satisfied.

We need to consider four cases.

Case 1: $n \equiv 1, 3, 7, 9 \pmod{10}$.

In this case, the necessary conditions are satisfied only for $p \equiv 1, 5 \pmod{10}$. Suppose $p \equiv 1, 5 \pmod{10}$. Using Lemma 4.2, an equitably 3-coloured 5-cycle decomposition of $K_{p(n)}$ can be constructed for any $n \equiv 1, 3, 7, 9 \pmod{10}$ from an equitably 3-coloured 5-cycle decomposition of $K_{p(1)}$, which exists by Lemma 4.3.

Case 2: $n \equiv 2, 4, 6, 8 \pmod{10}$.

In this case, the necessary conditions are satisfied only for $p \equiv 0, 1 \pmod 5$. Let $p \equiv 0, 1 \pmod 5$. Using Lemma 4.2, an equitably 3-coloured 5-cycle decomposition of $K_{p(n)}$ can be constructed for any $n \equiv 2, 4, 6, 8 \pmod{10}$ from an equitably 3-coloured 5-cycle decomposition of $K_{p(2)}$, which exists by Lemma 4.4.

Case 3: $n \equiv 5 \pmod{10}$.

When $n \equiv 5 \pmod{10}$, the necessary conditions are satisfied for any odd positive integer p. Let p be an odd, positive number. By Lemma 4.2, an equitably 3-coloured 5-cycle decomposition of $K_{p(n)}$ can be constructed for any $n \equiv 5 \pmod{10}$ from an equitably 3-coloured 5-cycle decomposition of $K_{p(5)}$, which exists by Lemma 4.11.

Case 4: $n \equiv 0 \pmod{10}$.

When $n \equiv 0 \pmod{10}$, the necessary conditions are satisfied for any positive integer p such that $p \neq 2$. Let p be a positive integer such that $p \neq 2$. By Lemma 4.2, an equitably 3-coloured 5-cycle decomposition of $K_{p(n)}$ can be constructed for any $n \equiv 0 \pmod{10}$ from an equitably 3-coloured 5-cycle decomposition of $K_{p(10)}$, which exists by Lemma 4.12. □

5. Some concluding remarks

Some interesting problems arise from this work. For example, it may be interesting to consider equitably 3-colourable 4-cycle decompositions of complete multipartite graphs in which the parts are not all of the same size. Furthermore, what restrictions (if any) are there on equitably 4-colourable m-cycle decompositions of complete equipartite graphs, for small values of m?

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